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An approach to multipole radiation through Green's dyadic

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Abstract. The article discusses a conversion identity through which the series expansion of the Green's function pertaining to the scalar Helmholtz equation can be readily converted into a series expanded Green's dyadic. This expansion finds a simple application in the electromagnetic radiation theory leading to electric multipole and magnetic multipole terms of the vector potential and the resulting $\boldsymbol{E}$ and $\boldsymbol{B}$ fields. The dyadic is also used to obtain a series expansion of the magnetostatic vector potential. Finally, the role played by localized longitudinal and localized transverse currents in the generation of the electromagnetic field is briefly examined, leading to the conclusion that a localized longitudinal current is self-screening. That is, it does not produce any electromagnetic field outside the domain of its distribution.

1. Introduction

The most familiar treatment of the classical multipole radiation field [1, 2] usually follows a method suggested by Bouwkamp and Casimir [3]. In this method, the $\boldsymbol{E}$ and $\boldsymbol{B}$ fields are expanded into multipole terms by first solving the inhomogeneous Helmholtz equation for the *radial components* of these fields. The source densities in these equations appear somewhat cumbersome, making it difficult to get a transparent picture of the sources involved. Secondly, for many applications, particularly in Lagrangian formulations and quantum theories, the relevant fields are the scalar and vector potentials $(\Phi, \boldsymbol{A})$. It is, therefore, profitable to stress alternative methods which follow the standard technique of solving Maxwell's equations through an integral solution of the inhomogeneous Helmholtz equations satisfied by Φ and $\boldsymbol{A}$. Expansion of these potentials in terms of spherical harmonics leads to electric and magnetic multipole terms of the radiation field. In this article, we outline one such approach which centres around Green's dyadic and its series expansion.

The direct integral approach and subsequent multipole expansion with the help of Debye potentials can be found in several earlier papers [4, 5]. Shore and Menzel [6] have used vector spherical harmonics to a limited extent in a formalism worked in the Coulomb gauge. Morse and Feshbach [7] have outlined the Green's dyadic approach for solving vector Poisson's equations and the potentials used by them are essentially Debye potentials. Green's dyadic of a different sort, generating the $\boldsymbol{E}$ and $\boldsymbol{H}$ fields directly, are sometimes used by communication engineers [8].

The approach we propose in this article arrives at an expansion for $\boldsymbol{A}$ in the Lorentz gauge, employing the full set of vector spherical harmonics. This expansion, we hope, will be found to be more succintly expressed and easier to handle. The basic tool in our approach is a conversion identity that transforms the all-too-familiar Green's function into a Green's dyadic.

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In section 2, we have outlined this conversion mechanism and have demonstrated how the resulting series expansion of the Green's dyadic leads to a series solution of the vector Helmholtz equation. Section 3 treats the electromagnetic vector potential as a special case. The series expansion of the vector potential can be decomposed into two subseries, generating *magnetic* multipole components (shown in section 4) and *electric* multipole components (shown in section 5). In section 6, we have applied the Green's dyadic approach to obtain the multipole expansion for the *magnetostatic* field. In section 7, we have tried to gain some insight into the nature of *localized* longitudinal and transverse currents and the fields they generate. We have concluded that a *localized longitudinal* current is necessarily self-screening, implying thereby that it cannot produce any electromagnetic field outside the domain of its distribution. Section 8 gives a brief summary.

2. Converting the Green's function to a Green's dyadic

The scalar Helmholtz equation

$$(\nabla^2+k^2)\psi(\boldsymbol{r})=-f(\boldsymbol{r}) \tag{1}$$

admits the well known Green's function [1, pp 223–4, 742]

$$G(\boldsymbol{r},\boldsymbol{r}')=\frac{\exp(\mathrm{i}k|\boldsymbol{r}-\boldsymbol{r}'|)}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \tag{2}$$

which can be expanded into the series:

$$G(\boldsymbol{r},\boldsymbol{r}')=\mathrm{i}k\sum_{l=0}^{\infty}h_l(kr)j_l(kr')\sum_{m=-l}^{l}Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi'). \tag{3}$$

In the above equation (r,θ,ϕ) and (r',θ,ϕ') are the spherical polar coordinates of the radius vectors $\boldsymbol{r}$ and $\boldsymbol{r}'$, with the stipulation that $r>r'$. h_l and j_l represent, respectively, the spherical Hankel function of the first type (representing outgoing wave) and the spherical Bessel function. We shall, henceforth, write Ω to mean (θ,ϕ).

Assuming that all the sources responsible for the field are contained within a volume V, the complete solution of equation (1) is expressed in the integral form:

$$\psi(\boldsymbol{r})=\int_V G(\boldsymbol{r},\boldsymbol{r}')f(\boldsymbol{r}')\,\mathrm{d}^3r' \tag{4}$$

where r is larger than the maximum value of r' in the integral.

A corollary of equation (4) follows to be the solution of the vector Helmholtz equation

$$(\nabla^2+k^2)\boldsymbol{\psi}(\boldsymbol{r})=-\boldsymbol{f}(\boldsymbol{r}). \tag{5}$$

Equation (5) represents three scalar equations, one each for the three *Cartesian* components of $\boldsymbol{\psi}$, namely

$$(\nabla^2+k^2)\psi_i(\boldsymbol{r})=-f_i(\boldsymbol{r}) \qquad i=x,y,z.$$

Here ψ_i and f_i are, respectively, the *Cartesian* components of the vector field $\boldsymbol{\psi}$ and the source density vector $\boldsymbol{f}$. Solutions for the individual component fields combine to yield the required vector field in the integral form

$$\boldsymbol{\psi}(\boldsymbol{r})=\int_V G(\boldsymbol{r},\boldsymbol{r}')\boldsymbol{f}(\boldsymbol{r}')\,\mathrm{d}^3r'=\int_V \mathbf{G}(\boldsymbol{r},\boldsymbol{r}')\cdot\boldsymbol{f}(\boldsymbol{r}')\,\mathrm{d}^3r' \tag{6}$$

where

$$\mathbf{G}(\boldsymbol{r}, \boldsymbol{r}') = G(\boldsymbol{r}, \boldsymbol{r}')\mathbf{1} \tag{7}$$

is the *Green's dyadic* and **1** is the unity dyadic or the idemfactor.

The expressions (6) and (7) are coordinate independent even though equation (5) has a meaning when expressed in terms of components in a Cartesian system. We can now use equation (3) to expand **G** in the spherical coordinate system.

$$\mathbf{G} = \mathrm{i}k \sum_{l=0}^{\infty} h_l(kr) j_l(kr') \sum_{m=-l}^{l} Y_{lm}(\Omega) Y_{lm}^*(\Omega')\mathbf{1}. \tag{8}$$

The above series can be converted into a more useful form with the help of *vector spherical harmonics*, as we shall show.

The *vector spherical harmonics* are obtained through tensor multiplication of the scalar spherical harmonics $\{Y_{lm}(\Omega)\}$ with the spherical base vectors $\{\boldsymbol{e}_m^{(1)}\}$ defined [6, p 268; 9, pp 103] as

$$\boldsymbol{e}_{\pm 1}^{(1)} \equiv \mp \frac{(\boldsymbol{e}_x \pm \mathrm{i}\boldsymbol{e}_y)}{2^{1/2}} \qquad \boldsymbol{e}_0^{(1)} \equiv \boldsymbol{e}_z. \tag{9}$$

Here $\boldsymbol{e}_x, \boldsymbol{e}_y, \boldsymbol{e}_z$ are the Cartesian base vectors $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$. The spherical base vectors form the components of a spherical tensor $\boldsymbol{e}^{(1)}$ of rank 1. We shall, henceforth, write the spherical harmonics as $Y_m^{(l)}$, where the superscript in parentheses acts as a reminder that we are considering a spherical tensor of rank l.

The vector spherical harmonics $\{\boldsymbol{T}_m^{(j)}(l; \Omega)\}$ are the $2j+1$ components of the spherical tensor $\boldsymbol{T}^{(j)}(l; \Omega)$ obtained [6, p 295; 9, p 106] as a tensor product of $Y^{(l)}(\Omega)$ with $\boldsymbol{e}^{(1)}$.

$$\begin{aligned} \boldsymbol{T}^{(j)}(l; \Omega) &\equiv [Y^{(l)}(\Omega) \otimes \boldsymbol{e}^{(1)}]^{(j)} \\ \boldsymbol{T}_m^{(j)}(l; \Omega) &\equiv \sum_{\mu=-1}^{1} C(l, 1, j; m-\mu, \mu, m) Y_{m-\mu}^{(l)}(\Omega) \boldsymbol{e}_\mu^{(1)}. \end{aligned} \tag{10}$$

In the above, the symbol C stands for Clebsch-Gordan coefficient. Explicit expressions for these vector spherical harmonics [6, pp 412-8] for $l \geqslant 1$ are:

$$\boldsymbol{T}_m^{(l)}(l; \Omega) = \boldsymbol{X}_m^{(l)}(\Omega) \equiv \frac{\boldsymbol{L} Y_m^{(l)}(\Omega)}{[l(l+1)]^{1/2}} \qquad \boldsymbol{L} \equiv \frac{1}{i} \boldsymbol{r} \times \nabla \tag{11a}$$

$$\boldsymbol{T}_m^{(l-1)}(l; \Omega) = \frac{r^{l+1}}{[l(2l-1)]^{1/2}} \nabla \left\{ \frac{Y_m^{(l-1)}(\Omega)}{r^l} \right\} \tag{11b}$$

$$\boldsymbol{T}_m^{(l+1)}(l; \Omega) = \frac{1}{[(l+1)(2l+3)]^{1/2}} \frac{1}{r^l} \nabla \{ r^{l+1} Y_m^{(l+1)}(\Omega) \} \tag{11c}$$

and, for $l = 0$:

$$T_m^{(1)}(0; \Omega) = \frac{1}{(4\pi)^{1/2}} \boldsymbol{e}_m^{(1)} \tag{12a}$$

whereas, $\boldsymbol{T}_m^{(0)}(0; \Omega)$ and $\boldsymbol{T}_m^{(-1)}(0; \Omega)$ are undefined. We shall formally set

$$\boldsymbol{T}_m^{(j)}(0; \Omega) \equiv 0 \qquad \text{for } j = 0, -1. \tag{12b}$$

Formulae (11*b*) and (11*c*) are a simple corollary of the gradient formula [9]. Formula (11*a*) can be verified by noting that the C-G coefficients suggested in equation (10) are, up to multiplicative constant, the same coefficients that appear in the recursion relations for $Y_m^{(l)}$ (see [1] p 743, equation (16.28)). The rest of our discussion is based on the following identity.

$$\sum_{m=-l}^{l} Y_m^{(l)}(\Omega) Y_m^{(l)*}(\Omega')\mathbf{1} = \sum_{j=l-1}^{l+1} \sum_{m=-j}^{j} \boldsymbol{T}_m^{(j)}(l;\Omega)\boldsymbol{T}_m^{(j)*}(l;\Omega'). \tag{13}$$

This identity is valid for all values of l including $l=0$, provided we adopt the definition (12). Equation (13) has been proved in the Appendix.

The identity (13) transforms the series expansion of $\mathbf{G}$ from the elementary form shown in equation (8) to one which can be useful in a variety of contexts.

$$\mathbf{G}(\boldsymbol{r},\boldsymbol{r}') = \mathrm{i}k \sum_{l=0}^{\infty} \sum_{j=l-1}^{l+1} \sum_{m=-j}^{j} \{h_l(kr)\boldsymbol{T}_m^{(j)}(l;\Omega)\}\{j_l(kr')\boldsymbol{T}_m^{(j)*}(l;\Omega')\} \qquad r>r'. \tag{14}$$

As a consequence, the series solution of the vector Helmholtz equation (5) outside the source distribution can be written compactly as:

$$\boldsymbol{\psi}(\boldsymbol{r}) = \mathrm{i}k \sum_{l=0}^{\infty} \sum_{j=l-1}^{l+1} \sum_{m=-j}^{j} S_m^{(j)l} h_l(kr)\boldsymbol{T}_m^{(j)}(l;\Omega) \qquad r>R \tag{15a}$$

where

$$S_m^{(j)l} \equiv \int_V j_l(kr)\boldsymbol{T}_m^{(j)*}(l;\Omega)\cdot\boldsymbol{f}(\boldsymbol{r})\,\mathrm{d}^3r \tag{15b}$$

and R is the radius of a sphere centred at the origin and enclosing the entire source domain V.

3. Multipole expansion of the vector potential in the electromagnetic radiation theory

It is sufficient to determine only the vector potential $\boldsymbol{A}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$ in the *Lorentz gauge* by solving the Helmholtz equation [1, pp 219–20]:

$$(\nabla^2 + k^2)\boldsymbol{A}(\boldsymbol{r}) = -\frac{4\pi\boldsymbol{J}(\boldsymbol{r})}{c}. \tag{16}$$

Here $\boldsymbol{J}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$ is the harmonically varying localized current density and $k=\omega/c$. The scalar potential $\Phi(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$ obtains from $\boldsymbol{A}(\boldsymbol{r})$ through the formula

$$\Phi(\boldsymbol{r}) = \frac{1}{\mathrm{i}\omega}\nabla\cdot\boldsymbol{A} \tag{17}$$

and hence, does not need any separate treatment.

The general result (15) and equation (16) suggest that we can write the vector potential in the following series form:

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{4\pi\mathrm{i}k}{c} \sum_{l=0}^{\infty} \sum_{j=l-1}^{l+1} \sum_{m=-j}^{j} S_m^{(j)l} h_l(kr)\boldsymbol{T}_m^{(j)}(l;\Omega) \tag{18a}$$

where

$$S_m^{(j)l} \equiv \int_V j_l(kr)\boldsymbol{T}_m^{(j)*}(l;\Omega)\cdot\boldsymbol{J}(\boldsymbol{r})\,\mathrm{d}^3r. \tag{18b}$$

4. Magnetic multipole radiation

The sub-series of equation (18), corresponding to $j=l$ gives the so-called 'magnetic multipole' components. Denoting magnetic multipole with M, the corresponding vector potential is

$$\boldsymbol{A}(M;\boldsymbol{r})=\frac{4\pi\mathrm{i}k}{c}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}S_m^{(l)}h_l(kr)\boldsymbol{X}_m^{(l)}(\Omega) \tag{19a}$$

with

$$\boldsymbol{S}_m^{(l)}\equiv\int_V j_l(kr)\boldsymbol{X}_m^{(l)*}(\Omega)\cdot\boldsymbol{J}(\boldsymbol{r})\,\mathrm{d}^3r. \tag{19b}$$

It is seen from (11a) that, for any arbitrary radial function $f(r)$,

$$\nabla\cdot\{f(r)\boldsymbol{X}_m^{(l)}(\Omega)\}=0. \tag{20}$$

Therefore, equation (17) suggests that $\Phi=0$. The $\boldsymbol{E}$ and $\boldsymbol{B}$ fields in the source free regions are now obtained.

$$\boldsymbol{E}(M;\boldsymbol{r})=-\frac{1}{c}\frac{\partial\boldsymbol{A}(M;\boldsymbol{r})}{\partial t}$$

$$=\sum_{l=1}^{\infty}\sum_{m=-l}^{l}a_M(l,m)h_l(kr)\boldsymbol{X}_m^{(l)}(\Omega)\qquad r>R \tag{21}$$

$$\boldsymbol{B}(M;\boldsymbol{r})=-\frac{\mathrm{i}}{k}\nabla\times\boldsymbol{E}(M;\boldsymbol{r})\qquad r>R$$

where

$$a_M(l,m)=-\frac{4\pi k^2}{c}S_m^{(l)}$$

$$=-\frac{4\pi k^2}{c}\int_v j_l(kr)\boldsymbol{X}_m^{(l)*}(\Omega)\cdot\boldsymbol{J}(\boldsymbol{r})\,\mathrm{d}^3r. \tag{22}$$

These coefficients are similar to the coefficients defined by Panofsky and Phillips [2] and coincide with the magnetic multipole coefficients defined by Jackson [1]. This can be verified by looking at the definition of $\boldsymbol{X}_m^{(l)}$ in equation (11a) and performing the integration by parts.

5. Electric multipole radiation

The 'electric multipole' contributions come from the $j=l+1$ and $j=l-1$ terms of (18). The corresponding vector potential outside the source is

$$\boldsymbol{A}(E;\boldsymbol{r})=\frac{4\pi\mathrm{i}k}{c}\sum_{l=0}^{\infty}\sum_{m}h_l(kr)\{S_m^{(l-1)l}\boldsymbol{T}_m^{(l-1)}(l;\Omega)+S_m^{(l+1)l}\boldsymbol{T}_m^{(l+1)}(l;\Omega)\}. \tag{23}$$

Computation of the magnetic field, by taking the curl of the above potential, is facilitated by the following identities.

$$\nabla\times[h_l(kr)\boldsymbol{T}_m^{(l-1)}(l;\Omega)]=\mathrm{i}k\left(\frac{l-1}{2l-1}\right)^{1/2}h_{l-1}(kr)\boldsymbol{X}_m^{(l-1)}(\Omega). \tag{24a}$$

$$\nabla\times[h_l(kr)\boldsymbol{T}_m^{(l+1)}(l;\Omega)]=-\mathrm{i}k\left(\frac{l+2}{2l+3}\right)^{1/2}h_{l+1}(kr)\boldsymbol{X}_m^{(l+1)}(\Omega). \tag{24b}$$

$$\begin{aligned}\nabla\times[j_l(kr)\boldsymbol{X}_m^{(l)}(\Omega)]&\\ =\mathrm{i}k\bigg[\left(\frac{l+1}{2l+1}\right)^{1/2}&j_{l-1}(kr)\boldsymbol{T}_m^{(l)}(l-1;\Omega)\\ -\left(\frac{l}{2l+1}\right)^{1/2}&j_{l+1}(kr)\boldsymbol{T}_m^{(l)}(l+1;\Omega)\bigg].\end{aligned} \tag{24c}$$

These identities can be derived by using the explicit forms of $\boldsymbol{T}_m^{(J)}(l;\Omega)$ given in equations (11) and the following recursion relations† applicable to any function $z_l(x)$ representing any one of spherical Bessel, spherical Neumann and spherical Hankel functions.

$$\begin{aligned}\left(\frac{\mathrm{d}}{\mathrm{d}x}+\frac{l+1}{x}\right)z_l(x)&=z_{l-1}(x)\\ \left(\frac{\mathrm{d}}{\mathrm{d}x}-\frac{l}{x}\right)z_l(x)&=-z_{l+1}(x).\end{aligned} \tag{25}$$

The curl of the vector potential of equation (23) yields

$$\begin{aligned}\boldsymbol{B}(E;\boldsymbol{r})&=\nabla\times\boldsymbol{A}(E;\boldsymbol{r})\\ &=\frac{4\pi k^2}{c}\sum_{l=0}^{\infty}\sum_m\bigg\{S_m^{(l+1)l}\left(\frac{l+2}{2l+3}\right)^{1/2}h_{l+1}(kr)\boldsymbol{X}_m^{(l+1)}(\Omega)\\ &\quad-S_m^{(l-1)l}\left(\frac{l-1}{2l-1}\right)^{1/2}h_{l-1}(kr)\boldsymbol{X}_m^{(l-1)}(\Omega)\bigg\}\\ &=\frac{4\pi k^2}{c}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left\{\left(\frac{l+1}{2l+1}\right)^{1/2}S_m^{(l)l-1}-\left(\frac{l}{2l+1}\right)^{1/2}S_m^{(l)l+1}\right\}h_l(kr)\boldsymbol{X}_m^{(l)}(\Omega).\end{aligned} \tag{26}$$

The expression within { } can be reduced, using the definition of $S_m^{(J)l}$ given in equations (18) and the identity (24c).

$$\begin{aligned}\{\cdot\}&=\int_V\mathrm{d}^3r\,\boldsymbol{J}(\boldsymbol{r})\cdot\bigg[\left(\frac{l+1}{2l+1}\right)^{1/2}j_{l-1}(kr)\boldsymbol{T}_m^{(l)*}(l-1;\Omega)\\ &\quad-\left(\frac{l}{2l+1}\right)^{1/2}j_{l+1}(kr)\boldsymbol{T}_m^{(l)*}(l+1;\Omega)\bigg]\\ &=\frac{\mathrm{i}}{k}\int_V\mathrm{d}^3r\,\boldsymbol{J}(\boldsymbol{r})\cdot\nabla\times[j_l(kr)\boldsymbol{X}_m^{(l)*}(\Omega)].\end{aligned} \tag{27}$$

† Our recursion relations (25) follow from p 741 equations (16.14) of [1]. Our equation (24c) is found in a slightly different form in [6] p 413.

Therefore,

$$\begin{aligned} \boldsymbol{B}(E;\boldsymbol{r}) &= \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_E(l,m)h_l(kr)\boldsymbol{X}_m^{(l)}(\Omega) \qquad r>R \\ \boldsymbol{E}(E;\boldsymbol{r}) &= \frac{\mathrm{i}}{k}\nabla\times\boldsymbol{B}(E;\boldsymbol{r}) \qquad r>R \end{aligned} \tag{28}$$

where

$$a_E(l,m)=\frac{4\pi\mathrm{i}k}{c}\int_V \mathrm{d}^3r\boldsymbol{J}(\boldsymbol{r})\cdot[\nabla\times j_l(jr)\boldsymbol{X}_m^{(l)*}(\Omega)]. \tag{29}$$

These coefficients look similar to the coefficients defined by Panofsky and Phillips [2]. Exploiting the relation

$$\nabla\cdot\boldsymbol{J}(\boldsymbol{r})=\mathrm{i}\omega\rho(\boldsymbol{r}) \tag{30}$$

where $\rho(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$ is the electric charge density, and the fact that $j_l(kr)Y_m^{(l)}(\Omega)$ satisfies the Helmholtz equation, the integral in (29) can be performed by parts. It then reduces to the same coefficients used by Jackson [1], which we quote here for subsequent use.

$$a_E(l,m)=\frac{4\pi k^2}{\mathrm{i}[l(l+1)]^{1/2}}\int_V Y_m^{(l)*}(\Omega)\left[\rho\frac{\mathrm{d}}{\mathrm{d}r}\{rj_l(kr)\}+\frac{\mathrm{i}k}{c}(\boldsymbol{r}\cdot\boldsymbol{J})j_l(kr)\right]\mathrm{d}^3r. \tag{31}$$

6. The magnetostatic case

The Green's dyadic (14) corresponding to the vector Helmholtz equation will fit into the vector Poisson's equation [1, p 176] (corresponding to $k=0$) by noting that

$$j_l(kr)\xrightarrow[k\to 0]{}\frac{(kr)^l}{(2l+1)!!}$$

$$h_l(kr)\xrightarrow[k\to 0]{}-\frac{\mathrm{i}(2l-1)!!}{(kr)^{l+1}}.$$

With this modification, the Green's dyadic of equation (14) transforms to the form:

$$\mathbf{G}(\boldsymbol{r},\boldsymbol{r}')=\sum_{l=0}^{\infty}\frac{1}{2l+1}\frac{r'^l}{r^{l+1}}\sum_{j=l-1}^{l+1}\sum_{m=-j}^{j}\boldsymbol{T}_m^{(j)}(l;\Omega)\boldsymbol{T}_m^{(j)*}(l;\Omega'). \tag{32}$$

Consequently, the vector potential is

$$\boldsymbol{A}(\boldsymbol{r})=\sum_{l=0}^{\infty}\frac{4\pi}{(2l+1)c}\sum_{j=l-1}^{l+1}\sum_{m=-j}^{j}S_m^{(j)l}\frac{\boldsymbol{T}_m^{(j)}(l;\Omega)}{r^{l+1}} \qquad r>R \tag{33a}$$

where

$$S_m^{(j)l}\equiv\int_V r^l\boldsymbol{T}_m^{(j)*}(l;\Omega)\cdot\boldsymbol{J}(\boldsymbol{r})\,\mathrm{d}^3r. \tag{33b}$$

The term $j=l+1$ drops out from the sum in (33a). To see this we refer to equation (11c), use the fact that $\nabla\cdot\boldsymbol{J}=0$, and perform the integration in (33) by parts.

$$S_m^{(l+1)l}=\frac{1}{[(l+1)(2l+3)]^{1/2}}\int_V\nabla\{r^{l+1}Y_m^{(l+1)*}(\Omega)\}\cdot\boldsymbol{J}(\boldsymbol{r})\,\mathrm{d}^3r=0.$$

The term $j=l-1$ in the sum (33) is unnecessary because

$$\nabla\times\left\{\frac{1}{r^{l+1}}\boldsymbol{T}_m^{(l-1)}(l;\Omega)\right\}=0$$

as can be seen from equation (11*b*). Consequently,

$$\boldsymbol{B}(\boldsymbol{r})=\nabla\times\boldsymbol{A}(\boldsymbol{r}) \tag{34}$$

where the above $\boldsymbol{A}$ is the effective vector potential having the expanded version

$$\boldsymbol{A}(\boldsymbol{r})=-\mathrm{i}\sum_{l=1}^{\infty}\sum_{m=-l}^{l}\left(\frac{l+1}{l}\right)^{1/2}\frac{4\pi M_{lm}}{2l+1}\frac{\boldsymbol{X}_m^{(l)}(\Omega)}{r^{l+1}}\qquad r>R \tag{35}$$

where

$$M_{lm}\equiv\frac{\mathrm{i}}{c}\left(\frac{l}{l+1}\right)^{1/2}\int_V r^l\boldsymbol{X}_m^{(l)*}(\Omega)\cdot\boldsymbol{J}(\boldsymbol{r})\,\mathrm{d}^3r$$

$$=-\frac{1}{c(l+1)}\int_V r^lY_m^{(l)*}(\Omega)\nabla\cdot(\boldsymbol{r}\times\boldsymbol{J})\,\mathrm{d}^3r$$

is the magnetic multipole moment of the stationary current distribution. The above expansion can be obtained in an alternative manner, by converting the magnetic scalar potential to a vector potential by a conversion rule [10].

7. Self screening charges and currents

Any given distribution of current $\boldsymbol{J}(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$ can always be decomposed [1, p 222] into a longitudinal or irrotational current $\boldsymbol{J}_l(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$ and a transverse or solenoidal current $\boldsymbol{J}_t(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$.

$$\boldsymbol{J}(\boldsymbol{r})=\boldsymbol{J}_l(\boldsymbol{r})+\boldsymbol{J}_t(\boldsymbol{r}) \tag{36}$$

where, by definition,

$$\nabla\times\boldsymbol{J}_l\equiv0\qquad\nabla\cdot\boldsymbol{J}_t\equiv0 \tag{37a}$$

so that

$$\nabla\times\boldsymbol{J}_t=\nabla\times\boldsymbol{J}\qquad\nabla\cdot\boldsymbol{J}_l=\nabla\cdot\boldsymbol{J}. \tag{37b}$$

It follows that $\boldsymbol{J}_l$ and $\boldsymbol{J}_t$ are obtainable from a scalar potential χ and a vector potential $\boldsymbol{P}$ satisfying

$$\begin{aligned}&\boldsymbol{J}_l(\boldsymbol{r})\equiv-\nabla\chi\qquad\nabla^2\chi=-\nabla\cdot\boldsymbol{J}\\&\boldsymbol{J}_t(\boldsymbol{r})\equiv\nabla\times\boldsymbol{P}\qquad\nabla\cdot\boldsymbol{P}\equiv0\qquad\nabla^2\boldsymbol{P}=-\nabla\times\boldsymbol{J}.\end{aligned} \tag{38}$$

It is then obvious that even when $\boldsymbol{J}$ is localized, its components $\boldsymbol{J}_l$ and $\boldsymbol{J}_t$ are not necessarily so.

However, one can imagine *localized* $\boldsymbol{J}_l$ and $\boldsymbol{J}_t$ separately and examine the kind of electromagnetic fields they can generate. We wish to point out the following features of these fields.

(i) A *localized longitudinal* current $\boldsymbol{J}_l(\boldsymbol{r})\,\mathrm{e}^{-\mathrm{i}\omega t}$ is necessarily self-screening. That is, it produces *neither* an *electric* multipole *nor* a *magnetic* multipole field.

(ii) A *localized transverse* current $J_t(r)\,e^{-i\omega t}$ which is also transverse to the radius vector, that is $J_t(r)\cdot r=0$, can produce only a *magnetic* multipole field.

The second conclusion is obvious due to equation (30). In this case, $\nabla\cdot J_t=0$, so that $\rho=0$. Therefore $a_E(l,m)=0$ according to equation (31).

The first conclusion will be shown to follow from the expressions for the coefficients $a_M(l,m)$ and $a_E(l,m)$ appearing in equations (22) and (29).

Proof. Let us consider the integral in the expression for $a_M(l,m)$ in equation (22) and recall the expression for $X_m^{(l)}(\Omega)$ in equation (11).

$$I_1\equiv\int_V j_l(kr)X_m^{(l)*}(\Omega)\cdot J_l(r)\,d^3r$$

$$=-1/i[l(l+1)]^{1/2}\int_V r\times\nabla\{j_l(kr)\,Y_m^{(l)*}(\Omega)\}\cdot J_l(r)\,d^3r.$$

The integrand $=r\cdot\nabla\{j_l(kr)\,Y_m^{(l)*}(\Omega)\}\times J_l(r)$

$$=r\cdot\nabla\times\{j_l(kr)\,Y_m^{(l)*}(\Omega)J_l(r)\}\qquad(\because\nabla\times J_l=0)$$

$$=-\nabla\cdot\{r\times j_l(kr)\,Y_m^{(l)*}(\Omega)J_l(r)\}\qquad(\because\nabla\times r=0).$$

Since $J_l(r)$ is localized inside V, it follows by the divergence theorem that $I_1=0$. Hence $a_M(l,m)=0$.

Next, we consider the integrand in the integral expression for $a_E(l,m)$ in (29).

$$\nabla\times\{j_l(kr)X_m^{(l)*}(\Omega)\}\cdot J_l(r)=\nabla\cdot\{j_l(kr)X_m^{(l)*}(\Omega)\times J_l(r)\}\qquad(\because\nabla\times J_l=0).$$

Again, by the divergence theorem, the integral is zero. Hence,

$$a_E(l,m)=0.$$

□

It would be of interest to visualize what kind of localized current density can be self-screening. From equations (38) and (30),

$$J_l(r)=-\nabla\chi(r)\qquad\nabla^2\chi(r)=-i\omega\rho(r).\tag{39}$$

These equations are reminders of the electrostatic field derivable from a potential Φ:

$$E(r)=-\nabla\Phi(r)\qquad\nabla^2\Phi(r)=-4\pi\rho(r).\tag{40}$$

A localized electric field can be realized by placing a distribution of charges inside a volume V whose boundary surface is kept at zero potential. An example is shown in figure 1(a) where a spherical zero potential surface S of radius R encloses a space V inside of which lies a point charge $+Q$. The localized electrostatic field E, which does not extent beyond $r=R$, is also shown.

Equations (39) and (40) suggest that we can get a localized alternating longitudinal current $J_l(r)\,e^{-i\omega t}$ associated with an alternating charge density $\rho(r)\,e^{-i\omega t}$ by converting a localized electrostatic field with the prescription:

$$E(r)\rightarrow\frac{4\pi}{i\omega}J_l(r)\qquad\rho(r)\rightarrow\rho(r).\tag{41}$$

It should be noted, however, that in the first case $E(r)$ and $\rho(r)$ are *time-independent real* static fields, whereas in the second case $J_l(r)$ and $\rho(r)$ are, in general, *complex*

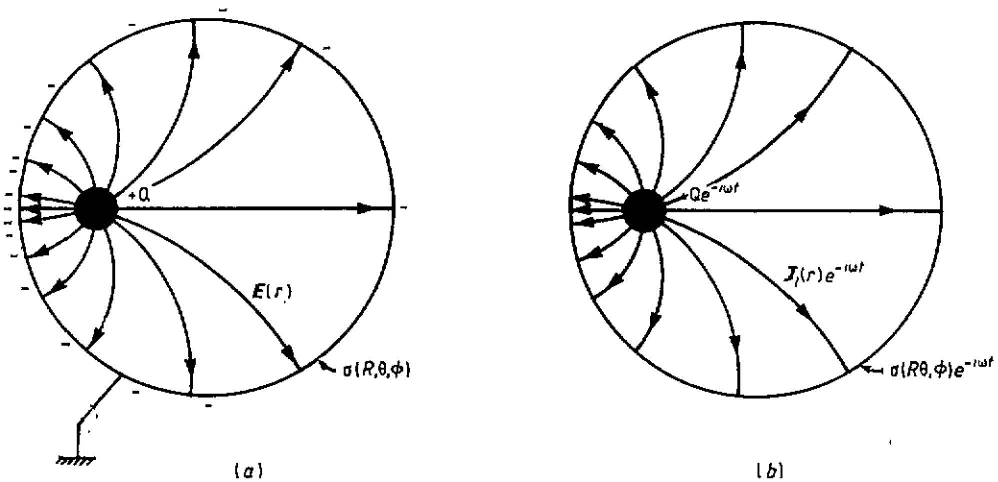

Figure 1. (a) Localized electrostatic field with $r=R$ surface at zero potential. σ represents surface charge density. (b) Localized longitudinal oscillating current obtained by converting figure 1(a). $\sigma(R, \theta, \phi)\,\mathrm{e}^{-\mathrm{i}\omega t}$ represents oscillating surface charge density.

amplitude functions of harmonically varying source densities. $J_l(r, t)$ and $\rho(r, t)$ differ by a phase of $\pi/2$, due to the presence of i in equation (41).

Figure (1b) shows the corresponding localized longitudinal current density obtained by the conversion mechanism (41). As already mentioned, such a localized current cannot produce any electromagnetic field beyond $r=R$.

8. Summary

The principal aim of this article has been to use Green's dyadic in obtaining a multipole expansion of the electromagnetic vector potential $A(r)\,\mathrm{e}^{-\mathrm{i}\omega t}$ resulting from a localized distribution of harmonically oscillating charges and currents. The steps used and the main conclusions reached are summarized as follows:

(i) The series expanded solution of the inhomogeneous scalar Helmholtz's equation involving scalar spherical harmonics is converted into the desired vector form (equation (15)) with the help of a derived identity (equation (13)) which converts the relevant Green's function into a Green's dyadic expanded in terms of vector spherical harmonics (equation (14)).

(ii) This vector series solution is utilized to obtain an expansion of the A field in the radiation theory (equation (18)), leading to a separation of the field into magnetic multipole terms corresponding to $j=l$ and the electric multipole terms corresponding to $j=l\pm1$ (equations (21), (22), (28), (29)).

(iii) The roles played by a purely longitudinal and a purely transverse localized current distribution are briefly examined in section 7, leading to the conclusion that the former does not produce any electromagnetic field and is, therefore, self-screening.

(iv) The expansion of the A field in equation (18) is made to suit vector Poisson's equation by letting $k\to0$, resulting in an expansion for the magnetostatic case (equation (35)).

Appendix

To prove the identity shown in equation (13).

We first write two minor identities which follow in a straightforward way from the definition of spherical base vectors, the definition of tensor product and the following Clebsch-Gordan coefficient [6, pp 275–95].

$$C(j, j, 0; m, -m, 0) = (-1)^{j+m}/(2j+1)^{1/2}. \tag{A.1}$$

These identities are:

$$\begin{aligned}[e^{(1)} \otimes e^{(1)}]^{(0)} &= -\{-e_1^{(1)} e_{-1}^{(1)} + e_0^{(1)} e_0^{(1)} - e_{-1}^{(1)} e_1^{(1)}\}/3^{1/2} \\ &= -(e_x e_x + e_y e_y + e_z e_z)/3^{1/2} \\ &= -1/3^{1/2} \mathbf{1}\end{aligned} \tag{A.2}$$

$$\begin{aligned}[Y^{(l)}(\Omega) \otimes Y^{(l)}(\Omega')]^{(0)} &= \sum_m \frac{(-1)^{l+m}}{(2l+1)^{1/2}} Y_m^{(l)}(\Omega) Y_{-m}^{(l)}(\Omega') \\ &= \frac{(-1)^l}{(2l+1)^{1/2}} \sum_m Y_m^{(l)}(\Omega) Y_m^{(l)*}(\Omega').\end{aligned} \tag{A.3}$$

It is then obvious that

$$\begin{aligned}&\sum_m Y_m^{(l)}(\Omega) Y_m^{(l)*}(\Omega') \mathbf{1} \\ &\quad = (-1)^{l+1} [3(2l+1)]^{1/2} [[Y^{(l)}(\Omega) \otimes Y^{(l)}(\Omega')]^{(0)} \otimes [e^{(1)} \otimes e^{(1)}]^{(0)}]^{(0)} \\ &\quad = (-1)^{l+1} [3(2l+1)]^{1/2} \sum_{j=l-1}^{l+1} (2j+1) \begin{Bmatrix} l & l & 0 \\ 1 & 1 & 0 \\ j & j & 0 \end{Bmatrix} \\ &\quad \times [[Y^{(l)}(\Omega) \otimes e^{(1)}]^{(j)} \otimes [Y^{(l)}(\Omega') \otimes e^{(1)}]^{(j)}]^{(0)}.\end{aligned}$$

The value of the above 9-j symbol can be obtained from Messiah [11]. Recalling the definition of $T_m^{(j)}(l; \Omega)$ from equation (10) and taking the Clebsch-Gordan coefficient from equation (A.1), the above identity now reduces to the following form.

$$\sum_{m=-l}^{l} Y_m^{(l)}(\Omega) Y_m^{(l)*}(\Omega') \mathbf{1} = (-1)^{l+1} \sum_{j=l-1}^{l+1} \sum_{m=-j}^{j} (-1)^{j+m} T_m^{(j)}(l; \Omega) T_{-m}^{(j)}(l; \Omega'). \tag{A.5}$$

It is seen from equation (11) that

$$\begin{aligned}T_{-m}^{(l\pm1)}(l; \Omega) &= (-1)^m T_m^{(l\pm1)*}(l; \Omega) \\ T_{-m}^{(l)}(l; \Omega) &= (-1)^{m+1} T_m^{(l)*}(l; \Omega).\end{aligned} \tag{A.6}$$

Substitution in (A.5) now leads to the identity (13) for $l \geq 1$.

Now we consider the special case corresponding to $l = 0$. Using the expressions for $T_m^{(j)}(0, \Omega)$ given in equations (12) we notice that the right-hand side of equation (13) takes the form

$$\begin{aligned}\sum_{m=-1}^{1} T_m^{(1)}(0; \Omega) T_m^{(1)*}(0; \Omega) &= \frac{1}{4\pi} \sum_m e_m^{(1)} (-1)^m e_{-m}^{(1)} \\ &= \frac{1}{4\pi} \mathbf{1} \qquad \text{(cf equation (A.2))} \\ &= Y_0^{(0)}(\Omega) Y_0^{(0)*}(\Omega') \mathbf{1}.\end{aligned} \tag{A.7}$$

Hence the identity (13) is valid for $l = 0, 1, 2, 3, \ldots$.

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